

DOUBLE SOLIDS, CATEGORIES AND NON-RATIONALITY

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To Slava — teacher and friend — with admiration.

ABSTRACT. This paper suggests a new approach to questions of rationality of threefolds based on category theory. Following [BFK10] and [FKb] we enhance constructions from [Kuz09] by introducing Noether–Lefschetz spectra — an interplay between Orlov spectra [Ol94] and Hochschild homology. The main goal of this paper is to suggest a series of interesting examples where above techniques might apply. We start by constructing a sextic double solid X with 35 nodes and torsion in $H^3(X, \mathbb{Z})$. This is a novelty — after the classical example of Artin and Mumford (1972), this is the second example of a Fano threefold with a torsion in the 3-rd integer homology group. In particular X is non-rational. We consider other examples as well — V_{10} with 10 singular points and double covering of quadric ramified in octic with 20 nodal singular points.

After analyzing the geometry of their Landau–Ginzburg models we suggest a general non-rationality picture based on Homological Mirror Symmetry and category theory.

1. INTRODUCTION

This paper suggests a new approach to questions of rationality of threefolds based on category theory. It was inspired by recent work of V. Shokurov and by A. Kuznetsov’s idea about the Griffiths component (see [Kuz08]). This work is a natural continuation of ideas developed in [Ka09], [GKKN] and of ideas of Kawamata and his school.

We first extend classical example of Artin and Mumford to construct a sextic double solid X with 35 nodes and torsion in $H^3(X, \mathbb{Z})$. The construction is based on an approach by M. Gross and suggests close relation between Artin and Mumford example and the sextic double solid X with 35 nodes. This example, a novelty on its own, opens a possibility of series of interesting examples — V_{10} with 10 singular points and double covering of quadric ramified in octic with 20 nodal singular points.

In this paper we start investigating these examples from the point of view of Homological Mirror Symmetry (HMS). We consider the mirrors of the sextic double solid X with 35 nodes, of the Fano variety V_{10} with 10 singular points in general position and of the double covering of quadric ramified in octic with 20 nodal singular points. We note that the monodromy around the singular fiber over zero of the Landau–Ginzburg models is strictly unipotent in all these examples, which suggests that the categorical behavior should be very similar to the one of the Artin–Mumford example. We conjecture that the reason for categorical similarity in all these examples is that they contained the category of an Enriques surface as a semiorthogonal summand in their derived categories. This is done in Section 5, where we introduce Landau–Ginzburg models and compare their singularities.

In Section 6 we introduce several new rationality invariants coming out of the notions of spectra and enhanced Noether–Lefschetz spectra of categories. We give a conjectural

categorical explanation of the examples from Sections 2, 3, 4, 5. The novelty (conjecturally) is that non-rationality of these examples cannot be picked by Orlov spectra but it is detected by the Noether–Lefschetz spectrum.

The paper is organized as follows: in Sections 2, 3, 4 we describe classical calculations of a sextic double solid X with 35 nodes. Section 5 contains some mirror considerations studying some Landau–Ginzburg models. Section 6 suggests a general categorical framework for studying the phenomena in Sections 2–5.

The paper is based on examples we have analyzed in [KPb], [KNS], [FKa], [FKb], [KK]. All these suggest a direct connection between monodromy of Landau–Ginzburg models, spectra and wall crossings in the moduli space of stability conditions, which was partially explored in [IKS]. This paper is a humble attempt to shed some light on this connection. We expect that further application of this method will be the theory of three dimensional conic bundles — a small part of huge algebro–geometric heritage of V. Shokurov (see Remark 4.3). In particular we expect that Noether–Lefschetz spectra of categories would allow us to prove nonrationality of new classes of conic bundles — classes where the method of Intermediate Jacobian does not work.

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All varieties considered in this paper are defined over the field of complex numbers \mathbb{C} . The torsion subgroup of given group G is denoted by $Tors(G)$; the n -torsion subgroup is denoted by $Tors_n(G)$. We denote du Val singularities of ADE type by A_n , D_n , and E_n . We denote a Landau–Ginzburg model of a variety X by $LG(X)$.

2. DETERMINANTAL DOUBLE SOLIDS AND BRAUER–SEVERI VARIETIES

2.1. The classical Artin–Mumford example. A double solid is an irreducible double covering $\pi : X \rightarrow \mathbb{P}^3$. The branch locus of such π is a surface $S \subset \mathbb{P}^3$ of even degree. In 1972 Artin and Mumford gave an example of a special singular quartic double solid X (i.e. $\deg S = 4$) which is non-rational because of the existence of a non-zero 2-torsion in its integer cohomology group $H^3(X, \mathbb{Z})$, see [AM72]. Since quartic double solids are unirational (see, for instance, [IP99], Example 10.1.3(iii)), this gives (together with the examples presented at the same time by Iskovskikh–Manin and Clemens–Griffiths) an example of a non-rational unirational threefold.

In [AMG96] Aspinwall, Morrison and Gross present a special case of a singular Calabi–Yau threefold — an octic double solid X (i.e. $\deg S = 8$) with 80 nodes on S and a non-zero 2-torsion in $H^3(X, \mathbb{Z})$.

In this section we adapt an approach used in [AMG96] to check again the existence of the 2-torsion in $H^3(X, \mathbb{Z})$ for the Artin–Mumford quartic double solid X , and present an example of a sextic double solid X with 35 nodes and a non-zero torsion in $H^3(X, \mathbb{Z})$. In particular this special nodal sextic double solid is not rational. Other examples are presented in sections to follow.

2.2. Quadric bundles and determinantal double solids. Let X_0 be a smooth complex projective variety, let L be an invertible sheaf on X_0 , and let $E \rightarrow X_0$ be a vector bundle of rank $r \geq 2$ over X_0 .

A *quadric bundle* in E parameterized by L is an \mathcal{O}_{X_0} -map

$$\varphi: L^{-1} \rightarrow \text{Sym}^2 E^*.$$

The determinantal loci of φ are subvarieties

$$D_{r-k} = D_{r-k}(\varphi) = \{x \in X_0 : \text{rank } \varphi_x \leq r - k\}, \quad k = 0, 1, 2, \dots$$

Geometrically a quadric bundle φ represents a bundle of quadrics

$$\mathcal{Q} = \{Q_x \subset \mathbb{P}(E_x) : x \in X_0\},$$

and

$$D_{r-k} = \{x \in X_0 : \text{rank } Q_x \leq r - k\}.$$

If $D_{r-k} \subset X_0$ are nonempty and have the expected codimensions $k(k+1)/2$ then their classes in $A_*(X_0)$ can be computed by the formulas in [HT84] or [JLP82]. For our purposes we need only to know explicit formulas for first two determinantal D_{r-1} and D_{r-2} , which can be computed formally as follows. Rewrite φ in the form

$$\varphi: \mathcal{O}_{X_0} \rightarrow \text{Sym}^2(E^* \otimes L^{1/2}),$$

and compute $c(E^* \otimes L^{1/2}) = 1 + c_1 + c_2 + \dots + c_r$. Then

$$D_{r-1} = 2c_1 \quad \text{and} \quad D_{r-2} = 4(c_1 c_2 - c_3).$$

In particular case when the base $X_0 = \mathbb{P}^n$ is a projective space, then the determinantal locus D_{r-1} is a hypersurface in \mathbb{P}^n of even degree; therefore D_{r-1} defines a double covering

$$\pi: X \rightarrow \mathbb{P}^n$$

branched along D_{r-1} . We call such X a *determinantal double solid*.

2.3. Cohomological Brauer groups and Brauer–Severi varieties. Let X be a complex algebraic variety, let \mathcal{O}_X be the structure sheaf of X , and let \mathcal{O}_X^* be the sheaf of units in \mathcal{O}_X . The Picard group and the (cohomological) Brauer group of X are correspondingly the 1-st and the 2-nd cohomology groups

$$\text{Pic}(X) = H^1(X, \mathcal{O}_X^*) \quad \text{and} \quad \text{Br}(X) = H^2(X, \mathcal{O}_X^*).$$

There is an exact sequence

$$\text{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow H^2(X, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Br}(X) \rightarrow 0,$$

see 3.1 in part II of [Gr68]. If in addition X is non-singular and it fulfills conditions

$$(1) \quad \text{Pic}(X) = H^2(X, \mathbb{Z}) \quad \text{and} \quad H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0,$$

then by the universal coefficient theorem $\text{Br}(X) \cong \text{Tors}(H^3(X, \mathbb{Z}))$, see e.g. [AM72]. For any X as above, a *Brauer–Severi variety* over X is a variety \mathcal{P} with a structure of a \mathbb{P}^n -bundle $f: \mathcal{P} \rightarrow X$ over X .

Not any Brauer–Severi variety is a projectivisation of a vector bundle over X , and the Brauer group gives obstructions for a Brauer–Severi variety to be presented as a projectivisation of such. On X , we consider exact sequence

$$0 \rightarrow \mathcal{O}_X^* \rightarrow GL_{n+1} \rightarrow PGL_{n+1} \rightarrow 0,$$

where \mathcal{O}_X^* is the multiplicative group of X .

The corresponding long exact sequence is

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow H^1(X, GL_{n+1}) \xrightarrow{j} H^1(X, PGL_{n+1}) \xrightarrow{\delta} \mathrm{Br}(X) \longrightarrow \dots$$

The vector bundles $E \rightarrow X$ of rank $(n+1)$ are elements of the cohomology group $H^1(X, GL_{n+1})$, while the \mathbb{P}^n -bundles $\mathcal{P} \rightarrow X$ are elements of $H^1(X, PGL_{n+1})$.

Therefore by above sequence the \mathbb{P}^n -bundle \mathcal{P} is not a projectivisation of a vector bundle on X iff $\delta(\mathcal{P}) \neq 0$. Since $(n+1)\delta = 0$ then any \mathcal{P} with $\delta(\mathcal{P}) \neq 0$ gives rise to a non-zero $(n+1)$ -torsion element $\delta(\mathcal{P}) \in \mathrm{Br}(X)$. If moreover X fulfills conditions (1) then \mathcal{P} represents a non-zero $(n+1)$ -torsion element of $H^3(X, \mathbb{Z}) \cong \mathrm{Br}(X)$. In particular case we consider below \mathcal{P} is a \mathbb{P}^1 -bundle which is not a projectivisation of a vector bundle, thus representing a non-zero 2-torsion element of $H^3(X, \mathbb{Z})$.

In the next sections we will use the following:

Lemma 2.1. Torsion criterion for non-rationality. *For the smooth complex variety Y the torsion subgroup $\mathrm{Tors}(H^3(Y, \mathbb{Z}))$ is a birational invariant of X . In particular if Y is rational then $\mathrm{Tors}(H^3(Y, \mathbb{Z})) = 0$.*

Proof. See Proposition 1 in [AM72] or §9 in [Be83]. □

3. DETERMINANTAL SEXTIC DOUBLE SOLID X WITH A NON-ZERO 2-TORSION IN $H^3(X, \mathbb{Z})$

3.1. The double solids of Artin–Mumford, Aspinwall–Morrison–Gross, and determinantal sextic double solid. The Artin–Mumford threefold from [AM72] is a special double solid with a branch locus — a quartic surface S with 10 nodes and with a torsion in the 3-rd integer cohomology group $H^3 = H^3(\tilde{X}, \mathbb{Z})$, where $\tilde{X} \rightarrow X$ is the blowup of X at its nodes. As it was shown later by Endrass, the group H^3 of a double solid X branched over a nodal quartic surface S can have a non-zero torsion only in case when S has 10 nodes, see [En99]. Therefore the branch loci of eventual further examples of nodal 3-fold double solids with a non-zero torsion in the 3-rd integer cohomology group H^3 should be of degree d either equals 2 or ≥ 6 . If in addition we require such X to be a Fano threefold then d must be ≤ 6 , i.e. if exists such X must be a sextic double solid or a double quadric. Notice that non-singular Fano threefolds X have a zero torsion in $H^3 = H^3(X, \mathbb{Z})$, so the requirement X to be singular (and nodal — for simplicity) is substantial.

In [AMG96] Aspinwall, Morrison, and Gross study a special case of a Calabi–Yau threefold which is a double solid X with a torsion in H^3 and with a branch locus S of degree 8 (an octic double solid). The similarity between the Artin–Mumford quartic double solid and the octic double solid from [AMG96] is that they both are *determinantal* double solids. Both these varieties X are singular — in the Artin–Mumford case X has 10 ordinary double points (nodes) while the octic double solid from [AMG96] has 80 nodes.

Below we describe an example of a determinantal nodal sextic double solid X with a torsion in H^3 . After the example of Artin and Mumford, this is the 2-nd example of a (necessary) singular nodal Fano threefold (see above) with a torsion in the 3-rd integer cohomology group. In particular our X must be non-rational, see Lemma 2.1.

It is shown by Iskovskikh (see [Is80]) that the general sextic double solid is non-rational due to the small group $\mathrm{Bir}(X)$ of birational automorphisms of X . This argument has been extended later by Cheltsov and Park proving the non-rationality of certain singular sextic double solids, see [CP07].

From this point of view, the example studied below is a non-rational sextic double solid X with 35 ordinary double points. According to Cheltsov (private communication), the non-rationality of this X cannot be derived, at least for now, from the results of [CP07].

The proof of the non-rationality of X presented below follows ideas from Appendix in [AMG96].

3.2. The determinantal sextic double solid. Let $\mathbb{P}^3 \times \mathbb{P}^4 \subset \mathbb{P}^{19}$ be Segre variety of \mathbb{C}^* -classes of non-zero 4×5 matrices, and let

$$W = (\mathbb{P}^3 \times \mathbb{P}^4) \cap H \cap F$$

be a general complete intersection of $\mathbb{P}^3 \times \mathbb{P}^4$ with a hyperplane $H = \mathbb{P}^{18} \subset \mathbb{P}^{19}$ and a divisor F of bidegree $(1,2)$. Let $Z = (\mathbb{P}^3 \times \mathbb{P}^4) \cap H$, and denote by p_Z and p_W the restrictions of the projection $p: \mathbb{P}^3 \times \mathbb{P}^4 \rightarrow \mathbb{P}^3$ to Z and to W . The projection p_W defines a structure of a quadric bundle

$$p_W: W \rightarrow \mathbb{P}^3$$

on W with fibers — quadrics $Q_x = p_W^{-1}(x)$ in the 3-spaces

$$\mathbb{P}_x^3 = p_Z^{-1}(x) = (x \times \mathbb{P}^4) \cap H, \quad x \in \mathbb{P}^3.$$

The \mathbb{P}^3 -bundle $p_Z: Z \rightarrow \mathbb{P}^3$ is a projectivisation of the rank 4 vector bundle E on \mathbb{P}^3 defined by vanishing linear form H defining a hyperplane section h on fibers of $p: \mathbb{P}^3 \times \mathbb{P}^4 \rightarrow \mathbb{P}^3$:

$$0 \longrightarrow E \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{\otimes 5} \xrightarrow{h} \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0;$$

therefore $c(E^*) = 1 + h + h^2 + h^3$ in $A_*(\mathbb{P}^3) = \mathbb{C}[h]/(h^4)$. Since W is an intersection of $Z = \mathbb{P}(E) \rightarrow \mathbb{P}^3$ with a bidegree $(1,2)$ divisor, then the bundle of quadrics defining a quadric bundle $p_W: W \rightarrow \mathbb{P}^3$ is given by the map

$$\varphi: \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow S^2 E^*.$$

So $c(E^*(\frac{1}{2})) = 1 + c_1 + c_2 + c_3 = 1 + 3h + 4h^2 + \frac{13}{4}h^3$, and hence

$$[D_3(\varphi)] = 2c_1 = 6h \quad \text{and} \quad [D_2(\varphi)] = 4(c_1 c_2 - c_3) = 35.$$

For a general choice of a bidegree $(1,2)$ divisor F , the branch locus

$$S = D_3(\varphi)$$

is a sextic surface in \mathbb{P}^3 with 35 nodes — the 35 points of

$$\delta = D_2(\varphi) = \{p_1, \dots, p_{35}\}.$$

Let

$$\pi: X \rightarrow \mathbb{P}^3$$

be a double covering branched along sextic surface $S = D_3$. Since $\text{Sing}(S) = \delta$, and the points $p_i \in \delta$ are nodes of S , then sextic double solid X has 35 nodes — the preimages of the 35 points p_1, \dots, p_{35} of δ .

Proposition 3.1. *Let $W = (\mathbb{P}^3 \times \mathbb{P}^4) \cap H \cap F$ be a general complete intersection of $\mathbb{P}^3 \times \mathbb{P}^4$ with a hyperplane and a divisor of bidegree $(1,2)$, Then:*

(1) *the degeneration locus $S = D_3$ of quadric fibration $p_W: W \rightarrow \mathbb{P}^3$, induced by the projection $p: \mathbb{P}^3 \times \mathbb{P}^4 \rightarrow \mathbb{P}^3$, is a sextic surface with 35 nodes;*

(2) *Let $\pi: X \rightarrow \mathbb{P}^3$ be a double covering branched along the sextic surface $S = D_3$. Then the group $H^3(X, \mathbb{Z})$ contains a non-zero 2-torsion element; in particular X is non-rational.*

3.3. Proof of Proposition 3.1. Part (1) follows from previous considerations. It remains to verify (2). Following an approach from [AMG96], we will find below a non-zero 2-torsion element of $H^3(X, \mathbb{Z})$, by representing it as a Brauer–Severi variety over the smooth part of X . Together with Lemma 2.1 this completes the proof.

We consider quadric bundle $p_W: W \rightarrow \mathbb{P}^3 = \mathbb{P}^3(x)$, and restrict it over open subset

$$\mathbb{P}_0^3 = \mathbb{P}^3 - \delta.$$

We define

$$S_0 = S - \delta, \quad X_0 = X - \delta_X \quad \text{and} \quad W_0 = W - \delta_W,$$

where $\delta_X = \pi^{-1}(\delta)$ is isomorphic preimage of $\delta = \{p_1, \dots, p_{35}\}$ on X , and $\delta_W = p^{-1}(\delta)$ is the set of 35 rank 2 quadric surfaces $Q_i = p^{-1}(p_i)$, $i = 1, \dots, 35$. Outside δ_W , the projection p restricts to a quadric bundle

$$p_{W_0}: W_0 \rightarrow \mathbb{P}_0^3$$

with degeneration locus S_0 .

Let $\pi: X_0 \rightarrow \mathbb{P}_0^3$ be induced determinantal double covering branched along S_0 . As it follows from our construction the fibers of the quadric bundle $p_{W_0}: W_0 \rightarrow \mathbb{P}_0^3$ are quadrics $Q_x \subset \mathbb{P}_x^3, x \in \mathbb{P}^3 - \delta$.

Let \mathcal{P} be the family of lines $l \subset W_0$ in the quadrics $Q_x, x \in \mathbb{P}^3 - \delta$, and let $\mathcal{P}_0 \subset \mathcal{P}$ be the family of these lines $l \in \mathcal{P}$ which lie on quadrics $Q_x, x \in \mathbb{P}_0^3 - \delta$.

Let us denote by

$$f_P: \mathcal{P} \rightarrow \mathbb{P}^3$$

the map sending a line $l \subset Q_x$ to a point $x \in \mathbb{P}^3$, and let us denote by $f_P: \mathcal{P}_0 \rightarrow \mathbb{P}_0^3$ its restriction over \mathbb{P}_0^3 .

We also define

$$\pi_0: X_0 \rightarrow \mathbb{P}_0^3$$

to be the restriction of the double covering $\pi: X \rightarrow \mathbb{P}^3$ to $X_0 = X - \delta_X$.

For any point $x \in \mathbb{P}_0^3 - S_0 = \mathbb{P}^3 - S$ the quadric $Q_x \subset \mathbb{P}_x^3$ is smooth, while for any $x \in S_0 = S - \delta$ the quadric Q_x is a quadratic cone of rank 3 in \mathbb{P}_x^3 .

Then we have

$$f_P^{-1}(x) \cong P^1 \vee P^1 \quad \text{for } x \in \mathbb{P}_0^3 - S_0 = \mathbb{P}^3 - S,$$

$$\text{and } f_P^{-1}(x) \cong \mathbb{P}^1 \quad \text{for } x \in S_0 = S - \delta.$$

Since S_0 is also branch locus of the double covering $\pi_0: X_0 \rightarrow \mathbb{P}_0^3$, we identify points of X_0 with generators of quadrics $Q_x, x \in \mathbb{P}_0^3$. Therefore the mapping $\mathcal{P}_0 \rightarrow \mathbb{P}_0^3$ is represented as a composition

$$\mathcal{P}_0 \xrightarrow{f_0} X_0 \xrightarrow{\pi_0} \mathbb{P}_0^3,$$

where

$$f_0: \mathcal{P}_0 \rightarrow X_0$$

is a \mathbb{P}^1 -fibration sending the sets of lines l on the quadrics Q_x to the generators of Q_x containing l . Let

$$\tilde{X} \rightarrow X$$

be the blowup of X at 35 nodes of X identified with 35 double points p_1, \dots, p_{35} of the surface S . Following [AMG96] we see that \mathcal{P}_0 is not a projectivisation of a vector bundle over X_0 . This yields that the Brauer group $Br(\tilde{X})$ has a non-zero element of order two, representing a non-zero 2-torsion element in $H^3(\tilde{X}, \mathbb{Z})$.

Suppose that $f_0: \mathcal{P}_0 \rightarrow X_0$ is a projectivisation of a rank 2 vector bundle $E \rightarrow X_0$. Up to a twist by a line bundle, we can always assume that E has sections. Next, any section of E gives rise to a rational section of $f_0: \mathcal{P}_0 = \mathbb{P}(E) \rightarrow X_0$. The following lemma concludes the argument:

Lemma 3.2. *The \mathbb{P}^1 -fibration $f_0: \mathcal{P}_0 \rightarrow X_0$ has no rational sections. In particular \mathcal{P}_0 is not a projectivisation of a rank 2 vector bundle on X_0 .*

Proof (see [AMG96] for more detail). Suppose that f_0 has a rational section, i.e. a rational map $\sigma: X_0 \rightarrow \mathcal{P}_0$ defined over an open dense subset $U \subset X_0$ and such that $f_0(\sigma(u)) = u$ for any $u \in U$. By definition the points of \mathcal{P}_0 are the lines l that lie on the quadrics $Q_t, t \in \mathbb{P}_0^3$. Denote by $l_u \in \mathcal{P}_0$ the line $l_u = \sigma(u)$ for points $u \in U$, i.e.

$$\sigma: U \rightarrow \mathcal{P}_0, x \mapsto l_u.$$

Let $\pi: X \rightarrow \mathbb{P}^3$ be the double covering, and let $i: X \rightarrow X$ be the involution interchanging two possibly coincident π -preimages of the points $x \in \mathbb{P}^3$. Without any loss of generality (e.g. by replacing U by $U \cap i(U)$) we may assume that $U = i(U)$. Let $D \subset W$ be Zariski closure of set

$$\{l_u \cap l_{i(u)} : u \in U \text{ and } u \neq i(u)\}.$$

The variety D is a 3-fold in W that intersects the general quadric $Q_x \subset \mathbb{P}_x^3 = x \times \mathbb{P}^3$, $x = \pi(u)$ at a unique point — the point $y(u) = l_u \cap l_{i(u)}$, i.e. $DQ_x = 1$.

The 5-fold $W = (\mathbb{P}^3 \times \mathbb{P}^4) \cap H \cap (F(x; y) = 0)$ is an ample divisor in the 6-fold $Z = (\mathbb{P}^3 \times \mathbb{P}^3) \cap H$, which in turn is an ample divisor in $\mathbb{P}^3 \times \mathbb{P}^4$.

Then by Lefschetz hyperplane section Theorem the restriction map defines an isomorphisms

$$H^4(\mathbb{P}^3 \times \mathbb{P}^4, \mathbb{Z}) \rightarrow H^4(Z, \mathbb{Z}) \rightarrow H^4(W, \mathbb{Z}).$$

In particular, the codimension two subvariety $D \subset W$ is a restriction of a codimension two subvariety of $\mathbb{P}^3 \times \mathbb{P}^4$ to W .

In the Chow ring

$$A_*(\mathbb{P}^3 \times \mathbb{P}^4) = \mathbb{Z}[h_1, h_2]/(h_1^4 = h_2^5 = 0, h_1^3 h_2^4 = 1),$$

the class of the fibre Q_x of $p: W \rightarrow X$ is $2h_1^3 h_2^2$. Since codimension 2 cycles on $\mathbb{P}^3 \times \mathbb{P}^4$ are generated over \mathbb{Z} by h_1^2, h_2^2 , and $h_1 h_2$, then the intersection number of any codimension 2 cycle on W with general quadric Q_x is even, which contradicts equality $DQ_x = 1$. \square

Notice also that the varieties X_0 and \tilde{X} fulfill conditions (1) from 2.3, so $Br(X_0)$ and $Br(\tilde{X})$ are isomorphic to $H^3(X_0, \mathbb{Z})$ and $H^3(\tilde{X}, \mathbb{Z})$.

Theorem 3.3. *The \mathbb{P}^1 -bundle \mathcal{P}_0 represents a non-zero 2-torsion element in $Br(X) = H^3(\tilde{X}, \mathbb{Z})$. In particular, \tilde{X} and hence X is non-rational.*

Proof. Let $E_i, i = 1, \dots, 35$ be the exceptional divisors of the blowup $\tilde{X} \rightarrow X$ at the nodes p_1, \dots, p_{35} . Then by [Gr68], for the Brauer groups of $X_0 = X - \{p_1, \dots, p_{35}\} \cong \tilde{X} - \cup\{E_i : i = 1, \dots, 35\}$ there is an exact sequence

$$0 \rightarrow Br(\tilde{X}) \rightarrow Br(X_0) \rightarrow \bigoplus_{i=1}^{35} H^1(E_i, \mathbb{Q}/\mathbb{Z}),$$

and since for surfaces $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ one has $H^1(E_i, \mathbb{Q}/\mathbb{Z}) = 0, i = 1, \dots, 35$, then $Br(\tilde{X}) \cong Br(X_0)$. \square

It follows from Lemmas 3.2 and 2.3 that \mathcal{P}_0 represents a non-zero 2-torsion element of $H^3(\tilde{X}, \mathbb{Z})$. Combining with Lemma 2.1 we get non-rationality of \tilde{X} , and hence — *the non-rationality* of X . This proves Proposition 3.1.

4. ARTIN–MUMFORD QUARTIC DOUBLE SOLID

4.1. Quadrics in \mathbb{P}^3 and Artin–Mumford quartic double solid. Let $\mathbb{P}^3 = \mathbb{P}^3(y)$, $(y) = (y_0 : \dots : y_3)$ be the 3-dimensional complex projective space. In the space $\mathbb{P}^9 = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(2)))$ of quadrics in \mathbb{P}^3 regard the determinants

$$\Delta_1 \subset \Delta_2 \subset \Delta_3 \subset \mathbb{P}^9$$

where

$$\Delta_k = \{Q \in \mathbb{P}^9 : \text{rank } Q \leq k, k = 1, 2, 3\}.$$

The elements of \mathbb{P}^9 are \mathbb{C}^* -classes of symmetric 4×4 matrices $Q = (q_{ij})$, $0 \leq i, j \leq 3$, and the determinants Δ_k , $1 \leq k \leq 3$, defined by vanishing $(k+1) \times (k+1)$ minors of Q have the following properties; for more details see e.g. §1 in [Co83]:

$\Delta_3 \subset \mathbb{P}^9$ is a quartic hypersurface;

$\Delta_2 = \text{Sing } \Delta_3$ has dimension 6 and degree 10;

$\Delta_1 = \text{Sing } \Delta_2 = v_2(\mathbb{P}^3)$ is the Veronese image of \mathbb{P}^3 in \mathbb{P}^9 ;

The determinantal quartic Δ_3 has an ordinary double singularity along $\Delta_2 - \Delta_1$.

Consider general 3-space $\mathbb{P}^3 = \mathbb{P}^3(x) \subset \mathbb{P}^9$. As it follows from previous considerations:

$S = \mathbb{P}^3 \cap \Delta_3$ is a quartic surface with only singularities — the 10 points of intersection $\delta = \mathbb{P}^3 \cap \Delta_2 = \{p_1, \dots, p_{10}\}$, and any p_k , $i = 1, \dots, 10$, is an ordinary double point (a node) of S .¹

Since $\deg(S) = 4$ is an even number, there exists a double covering

$$\pi: X \rightarrow \mathbb{P}^3$$

branched along S , i.e. X is a determinantal quartic double solid.

The double solid X has 10 nodes — isomorphic preimages of the 10 nodes p_1, \dots, p_{10} of branch locus S , which we also denote by p_1, \dots, p_{10} . Let \tilde{X} be the blowup of X at these 10 points. In the same way as in Section 3 we get:

Proposition 4.1. *The group $H^3(\tilde{X}, \mathbb{Z})$ contains a non-zero 2-torsion element; in particular X it is non-rational.*

Remark 4.2. In [AM72], Artin and Mumford prove stronger result: $\text{Tors}(H^3(X, \mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$, by using splitting of discriminant curve for natural conic bundle structure on X , see also Theorem 2 in [Za77].

¹ The quartic surfaces defined as determinantal loci of 3-spaces of quadrics in projective 3-space appear in the works of A. Cayley since the 80's of 19-th century under the name *quartic symmetroids*.

4.2. Artin–Mumford quartic double solids and Enriques surfaces. We start by recalling well known connection between Artin–Mumford double solids and Enriques surfaces, defined by Reye congruences, see e.g. [Co83]. In above notation, the Artin–Mumford double solids are defined by the general 3-spaces $\mathbb{P}^3(x)$ in the space $\mathbb{P}^9 = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3(y)}(2)))$ of quadrics in $\mathbb{P}^3(y)$, $(y) = (y_0 : \dots : y_3)$. Let

$$\{Q_x\} = \{Q_x \subset \mathbb{P}^3(y) : x \in \mathbb{P}^3(x) = \mathbb{P}^3(x_0 : \dots : x_3)\}$$

be the set of quadrics in $\mathbb{P}^3(y)$ defined by the 3-space $\mathbb{P}^3(x)$. Let G be the Grassmannian of lines $l \subset \mathbb{P}^3(y)$.

It is known that general line $l \subset \mathbb{P}^3(y)$ lies on a unique quadric from the family $\{Q_x\}$, and the set of lines

$$R = \{l \in G : \text{the line } l \subset \mathbb{P}^3(y) \text{ lies in a } \mathbb{P}^1\text{-family of quadrics } Q_x\}$$

is an Enriques surface in $G = G(2, 4)$ called classically a *Reye congruence*, see [Co83]. Let τ be an involution

$$(x, y) \xleftrightarrow{\tau} (y, x)$$

on $\mathbb{P}^3(x) \times \mathbb{P}^3(y)$. The fixed point set of τ is the diagonal Δ defined by $\{x = y\}$ in $\mathbb{P}^3(x) \times \mathbb{P}^3(y)$. For a quadratic form

$$Q(y) = \sum_{0 \leq i, j \leq 3} q_{ij} y_i y_j, \quad q_{ij} = q_{ji},$$

let

$$B(x, y) = \sum_{0 \leq i, j \leq 3} q_{ij} x_i y_j$$

be its corresponding bilinear form. Then a basis $Q_0(y), \dots, Q_3(y)$ of $\mathbb{P}^3(x) \subset \mathbb{P}^9$ defines a quadruple of bilinear forms $B_0(x, y), \dots, B_3(x, y)$, and hence — a linear section

$$\tilde{S} = (\mathbb{P}^3(x) \times \mathbb{P}^3(y)) \cap H_0 \cap \dots \cap H_3$$

where $H_i = (B_i(x, y) = 0)$. For a general choice of

$$\mathbb{P}^3(x) = \langle Q_0, \dots, Q_3 \rangle$$

the set \tilde{S} is a smooth complete intersection of 4 hyperplane sections of $\mathbb{P}^3(x) \times \mathbb{P}^3(y)$, and hence \tilde{S} is a smooth K3 surface — *Steiner* K3 surface in 3-space of quadrics $\mathbb{P}^3(y)$. Since all B_i are invariant under the involution τ , then \tilde{S} is also invariant under τ , i.e. $\tau(\tilde{S}) = \tilde{S}$. Therefore τ restricts to an involution $\tau : \tilde{S} \rightarrow \tilde{S}$; and since for general $\mathbb{P}^3(x)$ the surface \tilde{S} does not intersect diagonal Δ we conclude τ is without fixed points on \tilde{S} . The K3 surface \tilde{S} has following properties (see [Co83], [Ol94]):

Let $\mathbb{P}^3(x)$ be a general 3-space of the 9-space \mathbb{P}^9 of quadrics in $\mathbb{P}^3(y)$, and let $S = D_3 \subset \mathbb{P}^3(x)$, $R \subset G(2, 4)$ and \tilde{S} be correspondingly the quartic symmetroid, the Enriques surface (the Reye congruence), and the Steiner K3 surface defined by $\mathbb{P}^3(x)$. Then:

- (i) \tilde{S} is the blowup of S at its 10 nodes $\delta = \{p_1, \dots, p_{10}\}$;
- (ii) $R \subset G = G(2, 4)$ is isomorphic to the quotient \tilde{S}/τ of \tilde{S} by the involution τ .

Let $\pi : X \rightarrow \mathbb{P}^3(x)$ be the Artin–Mumford double solid, defined by the general 3-space $\mathbb{P}^3(x) \subset \mathbb{P}^9$, let $G = G(1 : \mathbb{P}^3(y))$ be as above, and let

$$\tilde{G} = \{(x, l) \in \mathbb{P}^3(x) \times G : l \subset Q_x\}.$$

Then (see §9 in [Be83]):

(iii) $\tilde{G} = \mathcal{P}$ (see the proof of Proposition 3.1), and the projection $\tilde{G} \rightarrow G$, $(x, l) \mapsto l$ is a blowup of the Enriques surface $R \subset G = G(2, 4)$.

(iv) The projection $\sigma: \tilde{G} \rightarrow \mathbb{P}^3$, $(x, l) \mapsto x$ factorizes into

$$\tilde{G} \xrightarrow{f} X \xrightarrow{\pi} \mathbb{P}^3(x),$$

and the restriction $\tilde{G}_0 \rightarrow X_0$ of f over $X_0 \subset X$ coincides with the \mathbb{P}^1 -bundle $f_0: \mathcal{P}_0 \rightarrow X_0$:

$$\begin{array}{ccc} \mathcal{P}_0 & \subset & \mathcal{P} \cong \tilde{G} \xrightarrow{\sigma} G(2, 4) \supset R \\ f_0 \downarrow & & \downarrow f \\ X_0 & \subset & X \\ \pi_0 \downarrow & & \downarrow \pi \\ \mathbb{P}^3 & \subset & \mathbb{P}^3 \end{array}$$

4.3. The non-rationality of X by the Criterion 2.1 (see [Be83]). We observe that since $\sigma: \mathcal{P} = \tilde{G} \rightarrow G(2, 4)$ is a blowup of the surface R in the 4-fold $G(2, 4)$, then

$$\begin{aligned} H^4(\mathcal{P}, \mathbb{Z}) &= \sigma^* H^4(G(2, 4), \mathbb{Z}) \oplus \sigma^{-1} H^2(R, \mathbb{Z}) \\ &\cong H^4(G(2, 4), \mathbb{Z}) \oplus H^2(R, \mathbb{Z}). \end{aligned}$$

Furthermore since R is an Enriques surface, then $c_1(R) \in H^2(R, \mathbb{Z})$ is an element of order 2. Therefore $Z = \sigma^{-1} c_1(R)$ is an element of order 2 in $H^4(\mathcal{P}, \mathbb{Z})$. After restriction, we get an element $Z_0 \in H^4(\mathcal{P}_0, \mathbb{Z})$ of order 2.

Since $f_0: \mathcal{P}_0 \rightarrow X_0$ is a \mathbb{P}^1 -bundle, then all fibers of f_0 are isomorphic to 2-dimensional spheres S^2 . Therefore the integral cohomology of \mathcal{P}_0 and X_0 fit in the Gysin sequence for S^2 -fibration:

$$\begin{aligned} \dots &\longrightarrow H^3(\mathcal{P}_0, \mathbb{Z}) \longrightarrow H^1(X_0, \mathbb{Z}) \xrightarrow{e} H^4(X_0, \mathbb{Z}) \\ &\xrightarrow{f_0^*} H^4(\mathcal{P}_0, \mathbb{Z}) \xrightarrow{f_{0*}} H^2(X_0, \mathbb{Z}) \longrightarrow \dots \end{aligned}$$

Here e is the cup-product with the Euler class $e(f_0) \in H^3(X_0, \mathbb{Z})$ of f_0 , see Chapter III, §14 in [BT82] and 4.11 in [Hi78].

If $Im(e) \neq 0$ then any non-zero element of $Im(e) \in H^4(X_0, \mathbb{Z})$ is a 2-torsion element, since $2e = 0$, see Theorem 4.11.2 (I) in [Hi78].

In case when $Im(e) = 0$ then $Z_0 \in Tors_2(H^4(\mathcal{P}_0, \mathbb{Z}))$ must be an image $Z_0 = f_0^*(C_0)$ of an element $C_0 \in H^4(X_0, \mathbb{Z})$, since $Tors(H^2(X_0, \mathbb{Z})) = 0$ (see p. 30 in [Be83]). Since in this case f_0^* is an embedding and Z_0 is a non-zero 2-torsion element of $H^4(\mathcal{P}_0, \mathbb{Z})$, then C_0 is also a non-zero 2-torsion element of $H^4(X_0, \mathbb{Z})$.

Thus in both cases there exists a 2-torsion element $C_0 \in H^4(X_0, \mathbb{Z})$.

Let $\sigma_X: \tilde{X} \rightarrow X$ be the blowup of X at the 10 nodes p_1, \dots, p_{10} of X , and let $E_i = \sigma_X^{-1}(p_i) \cong \mathbb{P}^1 \times \mathbb{P}^1$ be the 10 exceptional divisors on \tilde{X} . Since \tilde{X} is isomorphic to a disjoint union of X_0 and E_i , $i = 1, \dots, 10$, and $H^3(E_i, \mathbb{Z}) = H^3(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}) = 0$, then $H^4(X_0, \mathbb{Z})$ is embedded isomorphically in $H^4(\tilde{X}, \mathbb{Z})$. In particular $C_0 \in H^4(X_0, \mathbb{Z})$ is embedded as an element C of order two in $H^4(\tilde{X}, \mathbb{Z})$.

Since for a smooth projective complex threefold \tilde{X} one has

$$\text{Tor}s(H^4(\tilde{X}, \mathbb{Z})) \cong \text{Tor}s(H^3(\tilde{X}, \mathbb{Z}))$$

(see §1 in [AM72]), the 2-torsion element $C \in H^4(\tilde{X}, \mathbb{Z})$ represents equally a 2-torsion element $Z_C \in H^3(\tilde{X}, \mathbb{Z})$. By the Criterion 2.1 the last yields that \tilde{X} (and hence X) is non-rational.

Remark 4.3. It was suggested to us by K. Shramov that methods of [AMG96] can be applied to double covering of quadric ramified in octic with 20 singular points. More precisely we consider a divisor of bidegree (1,2) in $Q \times \mathbb{P}^3$, where Q is a quadric threefold. In this case we get a quadric fibration given by a map $\mathcal{O}_Q(-1) \rightarrow S^2(E^*)$, where E is a trivial vector bundle of rank 4. We get a 2-torsion (and hence nonrationality) in a middle cohomology of a double quadric with 20 nodal singular points. Using the fact that double covering of quadric ramified in octic with 20 singular points is a degeneration of three dimensional quartic we will study its Landau–Ginzburg model in Section 5.

5. MIRROR SIDE

In this section we turn to Homological Mirror Symmetry in an attempt to show that phenomena observed in previous sections is a part of much more general scheme. We briefly outline in Figure 1 a schematic picture of classical Homological Mirror Symmetry, in a version relevant for our purpose. For more details see [Ka09].

In what follows we describe fiberwise compactifications of weak Landau–Ginzburg models of quartic double solid, Fano threefold V_{10} , and of sextic double solid (see [Prz09a]). We conjecture that these compactifications are Landau–Ginzburg models of the Artin–Mumford example, V_{10} , and sextic double solid correspondingly in the sense of HMS.

Throughout this section we use the following standard notations for blowup. Consider affine variety

$$\{F(x_1, \dots, x_n) = 0\} \subset \mathbb{A}(x_1, \dots, x_n).$$

We blow up affine space $\{x_1 = \dots = x_k = 0\}$. The blown up hypersurface is given by the system of equations

$$\begin{cases} F(x_1, \dots, x_n) = 0, \\ x_i x'_j = x'_i x_j, \quad 1 \leq i, j \leq k, \end{cases}$$

in

$$\mathbb{A}(x_1, \dots, x_n) \times \mathbb{P}(x'_1 : \dots : x'_n).$$

Consider local chart $x'_1 \neq 0$. We choose coordinates

$$x_1, x'_2, \dots, x'_k, x_{k+1}, \dots, x_n.$$

In these coordinates blown up variety is zero locus of polynomial given by division of

$$F(x_1, x_1 x'_2, \dots, x_1 x'_k, x_{k+1}, \dots, x_n)$$

by maximal possible power of x_1 . We use notations x'_i 's for coordinates in this local chart instead of x_i 's for simplicity. We denote this local chart by $x_1 \neq 0$.

We embed fiberwise above pencil in a projective space or product of projective spaces and then resolve singularities. By [Prz09b], Theorem 11, in codimension one the compactification does not depend on which way we do it.

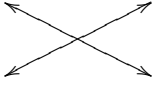
A-models (symplectic)	B-models (algebraic)
<p>$X = (X, \omega)$ a closed symplectic manifold</p> <p><i>Fukaya category</i> $\text{Fuk}(X)$. Objects are Lagrangian submanifolds L which may be equipped with flat line bundles. Morphisms are given by Floer cohomology $HF^*(L_0, L_1)$.</p>	<p>X a smooth projective variety</p> <p><i>Derived category</i> $D^b(X)$. Objects are complexes of coherent sheaves \mathcal{E}. Morphisms are $\text{Ext}^*(\mathcal{E}_0, \mathcal{E}_1)$.</p>
	
<p>A non-compact symplectic manifold Y with a proper map $W: Y \rightarrow \mathbb{C}$ which is a symplectic fibration with singularities.</p> <p><i>Fukaya–Seidel category of the Landau–Ginzburg model</i> $FS(LG(Y))$: Objects are Lagrangian submanifolds $L \subset Y$ which, at infinity, are fibered over $\mathbb{R}^+ \subset \mathbb{C}$. The morphisms are $HF^*(L_0^+, L_1)$, where the superscript $+$ indicates a perturbation removing intersection points at infinity.</p>	<p>Y a smooth quasi-projective variety with a proper holomorphic map $W: Y \rightarrow \mathbb{C}$.</p> <p>The category $D_{\text{sing}}^b(W)$ of algebraic B-branes which is obtained by considering the singular fibers $Y_z = W^{-1}(z)$, dividing $D^b(Y_z)$ by the subcategory of perfect complexes $\text{Perf}(Y_z)$, and then taking the direct sum over all such z.</p>

FIGURE 1. Classical Homological Mirror Symmetry.

5.1. The Landau–Ginzburg model of quartic double solid. The weak Landau–Ginzburg model for quartic double solid is given by

$$f = \frac{(x + y + 1)^4}{xyz} + z \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}].$$

We compactify pencil $\{f = \lambda, \lambda \in \mathbb{C}\}$, in the neighborhood of $\lambda = 0$ in $\mathbb{P}(x : y : z : t) \times \mathbb{A}(\lambda)$ and get hypersurface

$$\{(x + y + t)^4 + xyz(z - \lambda t) = 0\} \subset \mathbb{P}(x : y : z : t) \times \mathbb{A}(\lambda).$$

Its singularities are seven lines

$$\begin{aligned} l_0 &= \{x + y + t = z = \lambda = 0\}, & l_1 &= \{x = y = t = 0\}, & l_2 &= \{x + y = z = t = 0\}, \\ l_3 &= \{x = y + t = z = 0\}, & l_4 &= \{x = y + t = z + \lambda y = 0\}, & l_5 &= \{x + t = y = z = 0\}, \\ & & l_6 &= \{x + t = y = z + \lambda x = 0\}. \end{aligned}$$

Generically above singularities are locally products of du Val singularities of type A_3 by affine line. “Horizontal” lines l_2 – l_6 intersect “vertical” line l_0 ; moreover, pairs of lines l_3 and l_4 , l_5 and l_6 intersect l_0 at one point (see Figure 2).

We resolve singularities by blowing up these lines. At first we blow up the vertical line l_0 twice. After this the singularities are proper transforms of lines l_1 – l_6 and five lines lying

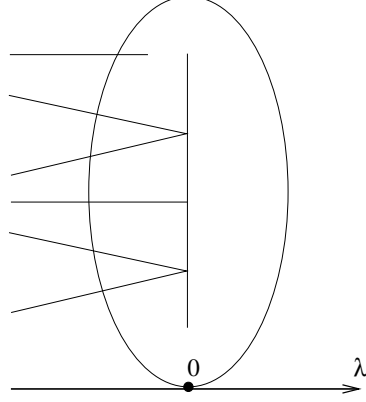


FIGURE 2. Singularities for quartic double solid.

on the exceptional divisors. Each of them intersect proper transform of one of lines l_2-l_6 . After blowing up these five lines we get threefold with six lines of singularities coming from l_1-l_6 which are of type A_3 along a horizontal affine line globally. Blowing them up fiberwise we get the final resolution. We carry this procedure in the following steps:

Step 0. The line l_1 is of type A_3 along affine line globally. Blowing it up twice we get horizontal exceptional fibers, so they do not give an additional component for fiber over $\lambda = 0$. We proceed resolution in the neighbourhood of line l_0 .

Step 1. Let $a = x + y + t$. Then our variety is given by

$$\{a^4 + xyz^2 = \lambda xyz(a - x - y)\} \subset \mathbb{P}(x : y : z : a) \times \mathbb{A}(\lambda)$$

and $l_0 = \{a = z = \lambda = 0\}$. There are two similar local charts: $x \neq 0$ and $y \neq 0$. Consider local chart $y \neq 0$. It contains lines of singularities l_0, l_2-l_4 . We study the resolution in this chart and double the picture over lines l_3, l_4 . In this local chart we have an affine hypersurface

$$a^4 + xz^2 = \lambda xz(a - x - 1)$$

and we need to blow up line $l_0 = \{a = z = \lambda = 0\}$.

The local chart 1a: $a \neq 0$. We have hypersurface

$$a^2 + xz^2 = \lambda xz(a - x - 1).$$

The exceptional divisor is given by equation $a = 0$, so it consists of three components

$$E_1^a = \{a = x = 0\}, \quad E_2^a = \{a = z + (x + 1)\lambda = 0\}, \quad E_3^a = \{a = z = 0\}.$$

The proper transform of the fiber over $\lambda = 0$ is $E_0 = \{\lambda = a^2 + xz^2 = 0\}$. The singularities are:

$$\begin{aligned} l_1^a &= \{x = z = a = 0\}, & l_2^a &= \{x = \lambda + z = a = 0\}, \\ l_3^a &= \{z = a = \lambda = 0\}, & l_4^a &= \{x + 1 = z = a = 0\}. \end{aligned}$$

We have:

$$E_2^a \cap E_3^a = l_3^a \cup l_4^a, \quad E_1^a \cap E_3^a = l_1^a, \quad E_0 \cap E_2^a \cap E_3^a = l_3^a.$$

All proper transforms of lines l_2-l_6 do not lie in this chart.

The local chart 1z: $z \neq 0$. There is nothing new in this chart: all we are interested in is contained in the chart **1a**.

The local chart 1 λ : $\lambda \neq 0$. We have hypersurface

$$\lambda^2 a^4 + xz^2 = xz(\lambda a - x - 1).$$

The exceptional divisor is given by equation $\lambda = 0$, so it consists of three components

$$E_1^\lambda = \{\lambda = x = 0\}, \quad E_2^\lambda = \{\lambda = z + x + 1 = 0\}, \quad E_3^\lambda = \{\lambda = z = 0\}.$$

The proper transform of fiber over $\lambda = 0$ does not lie in this chart. We have:

$$E_1^\lambda = E_1^a, \quad E_2^\lambda = E_2^a, \quad E_3^\lambda = E_3^a.$$

The singularities are:

$$\begin{aligned} l_1^\lambda &= \{x = z = \lambda = 0\} = E_1^\lambda \cap E_3^\lambda, \\ l_2^\lambda &= \{a = x = z = 0\} \quad \text{--- proper transform of } l_3, \\ l_3^\lambda &= \{x + 1 = z = \lambda = 0\} = E_2^\lambda \cap E_3^\lambda, \\ l_4^\lambda &= \{a = z = x + 1 = 0\} \quad \text{--- proper transform of } l_2, \\ l_5^\lambda &= \{x = z + 1 = \lambda = 0\} = E_1^\lambda \cap E_2^\lambda, \\ l_6^\lambda &= \{x = z + 1 = a = 0\} \quad \text{--- proper transform of } l_4. \end{aligned}$$

So, after first blow-up we get a configuration of components of central fiber drawn on Figure 3.

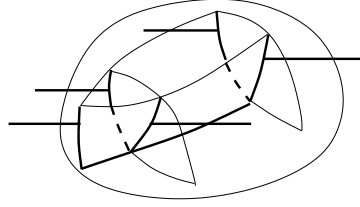


FIGURE 3. The picture after the first blowup.

Then we blow up the line l_3^a . It is enough to consider it in the chart **1a**. That is, we blow up the line

$$\{z = a = \lambda = 0\}$$

at

$$\{a^2 + xz^2 - \lambda xz(a - x - 1) = 0\}.$$

The only meaningful local chart is $\lambda \neq 0$. In this chart get the hypersurface

$$\{a^2 + xz^2 - xz(\lambda a - x - 1) = 0\}.$$

The exceptional divisor is

$$E^{a,\lambda} = \{\lambda = a^2 - xz(z + x + 1) = 0\}.$$

The singularities in its neighborhood are

$$\begin{aligned} \{a = x = z = 0\} &\quad \text{--- proper transform of } l_1^a, \\ \{a = x = z + 1 = 0\} &\quad \text{--- proper transform of } l_2^a, \\ \{a = x + 1 = z = 0\} &\quad \text{--- proper transform of } l_4^a. \end{aligned}$$

All of them lie on the exceptional divisor. So we did not get “new” singularities after this blowup. The divisors E_2^a , E_3^a now intersect only by proper transform of l_4^a ; the divisor E_1^a intersect E_2^a and E_3^a in two separated lines both intersect the line $E_1^a \cap E^{a,\lambda} = \{x =$

$a = \lambda = 0\}$. The proper image of E_0 intersects only $E^{a,\lambda}$ by a line lying far from the rest exceptional divisors.

Now we blow up line $l_4^a = l_3^\lambda$. The line l_3^a does not lie at the chart $\mathbf{1}\lambda$ so we can consider this blowup only in the chart $\mathbf{1}\lambda$. We make a change of variables $x \rightarrow x - 1$. Then we get a hypersurface

$$\{\lambda^2 a^4 + (x - 1)z^2 = (x - 1)z(\lambda a - x) = 0\}$$

and then we need to blow up the line

$$\{x = z = \lambda = 0\}.$$

We get one exceptional divisor, proper images of lines l_5^λ and l_6^λ that lie far from exceptional divisor, proper images of l_1^λ and l_2^λ (we will discuss them later) and a proper image of l_4^λ (in the other words, of l_2). It is globally of type A_3 along a line, so it resolves horizontally and does not give an exceptional divisors over $\lambda = 0$.

So, after this blowup the divisors E_2^λ and E_3^λ are separated.

Now we blow up the line l_1^λ . As before, we can do it in the chart $\mathbf{1}\lambda$. We have a hypersurface

$$\{\lambda^2 a^4 + xz(z + x + 1 - \lambda a) = 0\}$$

and need to blow up the line $\{x = z = \lambda = 0\}$.

We get proper transforms of l_3^λ and l_4^λ we already discussed, proper transforms of l_5^λ and l_6^λ we mention in the next paragraph, and a proper transform of l_2^λ (in the other words, of l_3). It is globally of type A_3 along a line, so it resolves horizontally and does not give an exceptional divisors in the central fiber.

Finally, the picture we get after blowing up the line l_5^λ is very similar to the picture we get after blowing up the line l_1^λ .

We summarize final picture of resolved singularities (see Figure 4).

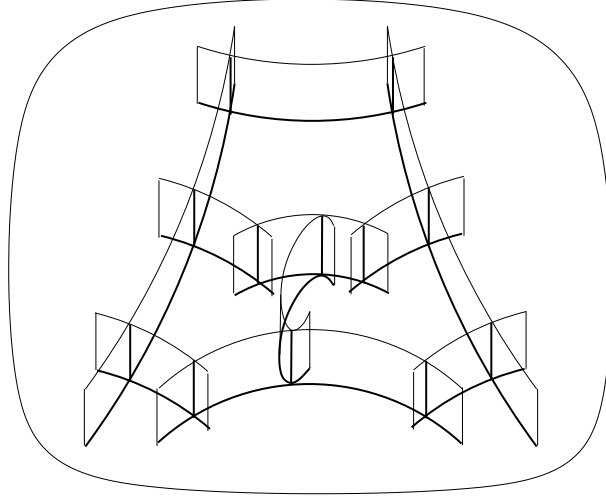


FIGURE 4. The fiber over 0 in Landau–Ginzburg model for a quartic double solid.

Via direct calculations (see [KPb], [KNS]) we get

Proposition 5.1. *The monodromy of the singular fiber at zero of the Landau–Ginzburg model for a quartic double solid with 10 singular points is strictly unipotent.*

The proof of above proposition is based on analysis of monodromy change under conifold transition.

5.2. The Landau–Ginzburg model of V_{10} . The weak Landau–Ginzburg model for a Fano variety V_{10} is

$$f = \frac{(x^2 + x + y + z + xy + xz + yz)^2}{xyz} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}].$$

Compactifying the pencil $\{f = \lambda, \lambda \in \mathbb{C}\}$, in the neighborhood of $\lambda = 0$ in $\mathbb{P}(x : y : z : t) \times \mathbb{A}(\lambda)$ we get a hypersurface

$$\{(x^2 + xt + zt + xz + yt + yz + xy)^2 = \lambda xyz t\} \subset \mathbb{P}(x : y : z : t) \times \mathbb{A}(\lambda).$$

Its singularities are twelve lines

$$l_1 = \{x + z = t = \lambda = 0\}, \quad l_2 = \{x = z = t = 0\}, \quad l_3 = \{x + z = y = t = 0\},$$

$$l_4 = \{x + y = t = \lambda = 0\}, \quad l_5 = \{x = y = t = 0\}, \quad l_6 = \{x + y = z = t = 0\},$$

$$l_7 = \{x = y = z = 0\}, \quad l_8 = \{x + z = y = \lambda = 0\}, \quad l_9 = \{x + t = y = z = 0\},$$

$$l_{10} = \{x = y + t = z = 0\}, \quad l_{11} = \{x + t = y = \lambda = 0\}, \quad l_{12} = \{x + t = z = \lambda = 0\},$$

and a conic

$$C = \{x = yt + zt + yz = \lambda = 0\}$$

(see Figure 5).

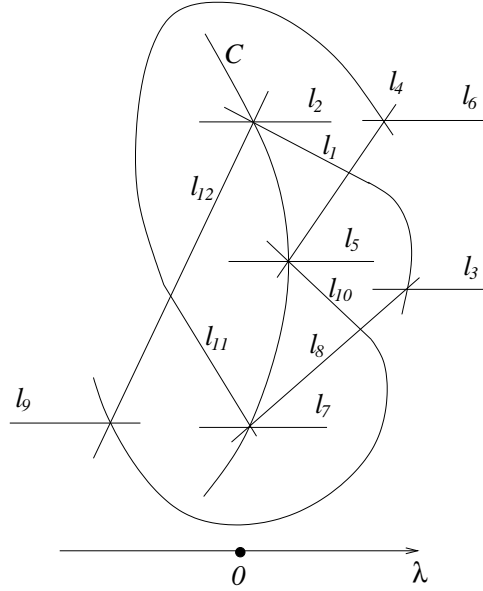


FIGURE 5. Singularities for V_{10} .

There is a symmetry $x \leftrightarrow y \leftrightarrow z$, so we have three types of singular lines: two horizontal line types and one vertical line type.

We blow up l_6 , we put $a = x + y$, and consider a local chart $x = 1$. In this chart coordinates of our family can be written as

$$\{(a + at + zt + az)^2 = \lambda(a - 1)zt\}.$$

In the neighborhood of l_6 it is analytically equivalent to a hypersurface $\{a^2 = \lambda zt\}$. In this local chart l_6 , l_{11} , and l_4 are given by equations $a = z = t = 0$, $a = z = \lambda = 0$, and

$a = t = \lambda = 0$ correspondingly. They are intersecting transversally lines of singularities of type A_1 . So, blowing l_6 up we get one horizontal exceptional divisor. In its neighborhood the singularities (proper images of l_{11} and l_4) are lines of singularities of type A_1 . Similarly, by symmetry, the same holds in a neighborhood of lines l_3 and l_9 . After blowups performed above the singularities can be seen on Figure 6.

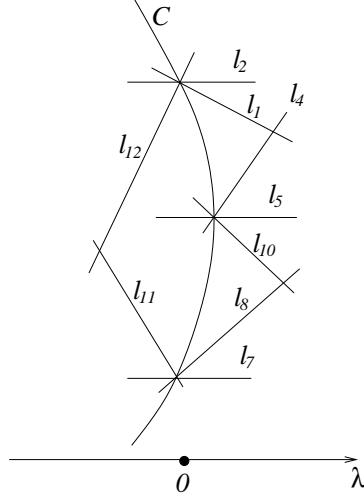


FIGURE 6. Singularities for V_{10} .

Let us blow up l_7 in local chart $t = 1$. We have a hypersurface

$$\{(x^2 + x + z + xz + y + yz + xy)^2 = \lambda xyz\}.$$

Analytically, in a neighborhood of l_7 it is isomorphic to a hypersurface

$$\{(x + y + z + yz)^2 = \lambda xyz\}.$$

The lines l_7 , l_8 , l_{11} and C are given by equations $x = y = z = 0$, $x + z = y = \lambda = 0$, $x + y = z = \lambda = 0$, and $y + z + yz = x = \lambda = 0$ correspondingly.

Consider a local chart $x \neq 0$ in the above blowup. We get a hypersurface

$$\{(1 + y + z + xyz)^2 = \lambda xyz\}.$$

The exceptional divisor is given by

$$\{x = y + z + 1 = 0\}$$

and singularities in its neighborhood are given by

$$l_1^x = \{x = y = z + 1 = 0\}, \quad l_2^x = \{x = y + 1 = z = 0\}, \quad l^x = \{x = y + z + 1 = \lambda = 0\}, \\ l_8 = \{y = z + 1 = \lambda = 0\}, \quad l_{11} = \{y + 1 = z = \lambda = 0\}.$$

In neighborhood of l_1^x the singularities are three intersecting lines: one horizontal l_1^x and two vertical lines l^x and l_8 . They are analytically equivalent to singular lines on the hypersurface $\{a^2 = \lambda xy\}$. We blow up l_1^x first and then l^x and l_8 . We get two non-intersecting exceptional divisors in the central fiber coming from l^x and l_8 .

Consider now a local chart $y \neq 0$ in the blowup. We get a hypersurface

$$\{(x + 1 + z + yz)^2 = \lambda xyz\}.$$

The exceptional divisor is given by

$$\{y = x + z + 1 = 0\}$$

and singularities in its neighborhood are given by

$$l^x = \{y = x + z + 1 = \lambda = 0\}, \quad l^y = \{x = y = z + 1 = 0\}, \quad l_2^x = \{y = z = x + 1 = 0\},$$

$$l_{11} = \{x + 1 = z = \lambda = 0\}, \quad C = \{x = 1 + z + yz = \lambda = 0\}.$$

In neighborhood of l^y the singularities form three intersecting lines of ordinary double points l^y , l_{11} , and C as before so we can resolve them in similar way.

Finally, we repeat with no change procedure in the last local chart $z \neq 0$. The lines l_1 and l_4 intersects transversally and are of type A_1 . Blowing them up one-by-one, we get in the central fiber two exceptional divisors intersecting in a line.

The central fiber of resolution shown on Figure 7. There are eleven surfaces.

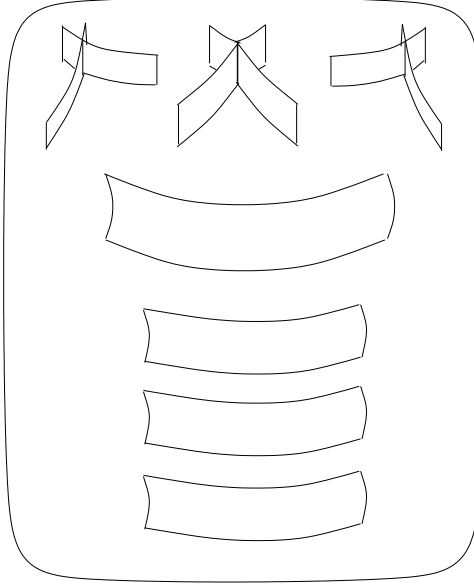


FIGURE 7. Singularities for V_{10} .

As before direct calculations based on [KPb], [KNS] give:

Proposition 5.2. *The monodromy of the singular fiber at zero of the Landau–Ginzburg model for V_{10} with 10 singular points is strictly unipotent.*

5.3. The Landau–Ginzburg model of sextic double solid. The weak Landau–Ginzburg model for a sextic double solid is

$$\frac{(x + y + z + 1)^6}{xyz} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}].$$

We are compactifying it in a projective space. The singularities are drawn on Figure 8. They are three vertical lines, three horizontal lines and a horizontal plane (lines are symmetric with respect to changing coordinates $x \leftrightarrow y \leftrightarrow z$).

We normalize the plane of singularities blowing it up (twice). Then we resolve horizontal vertical singularities. We record the structure of central fiber and vertical singularities on Figures 9, 11, and 10 glued in a way given by Figure 12.

The lines on Figure 9 are surfaces (we look on them “from above”). Bold ones intersect the “base” surface. The rectangle is a surface lying “over” the “base”. It intersects in two curves (which do not intersect the base surface) two remaining surfaces. The point

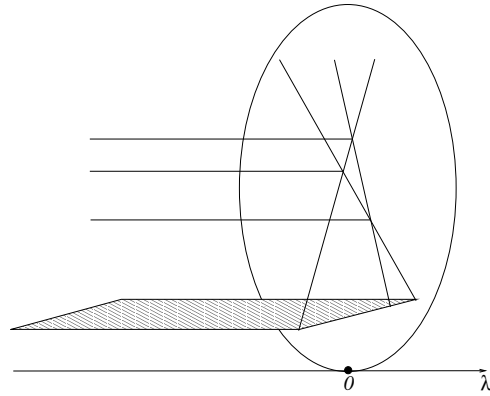


FIGURE 8. The singularities for sextic double solid.

of intersection of these lines and a “vertical” line of intersection of two other planes is denoted by a fat point. At the end we get nine surfaces and twelve lines recorded on the picture.

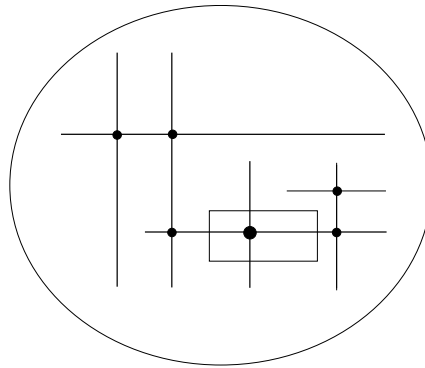


FIGURE 9. The final picture after a resolution of singularities in the neighborhood of non-normality locus.

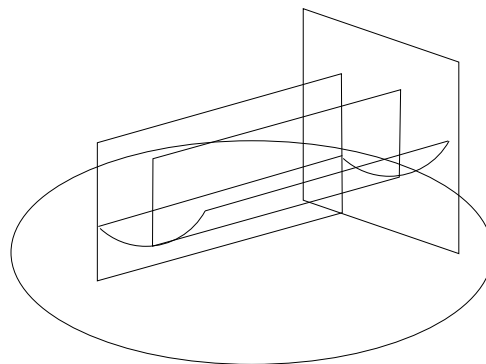


FIGURE 10. The resolution in the neighborhood of the “first sheet” of Figure 12.

We follow procedure of resolving singularities as in previous examples. The final picture is obtained by gluing configurations of surfaces drawn on Figures 9, 11, and 10 along Figure 12. More detailed description of Landau–Ginzburg model for sextic double solid see in [KPa]. Direct calculations ([KPb], [KNS]) produce.

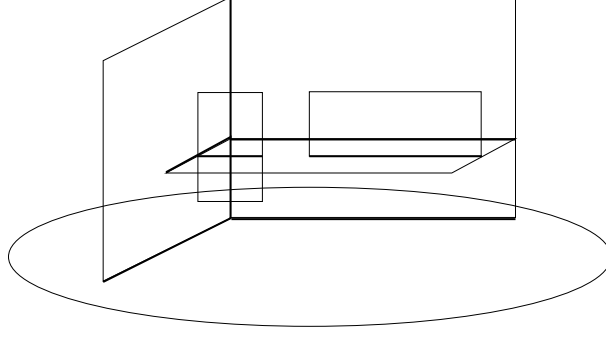


FIGURE 11. The resolution in the neighborhood of the “deep sheet” of Figure 12.

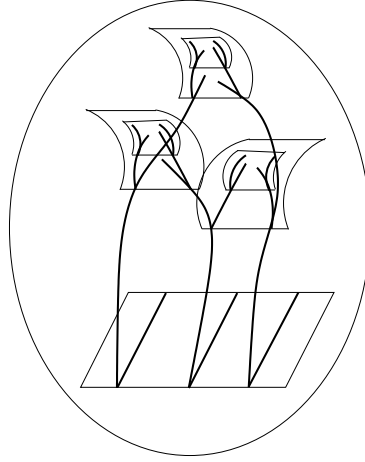


FIGURE 12. After blowing up of horizontal singularities.

Proposition 5.3. *The monodromy of the singular fiber at zero of the Landau–Ginzburg model for a sextic double solid with 35 singular points is strictly unipotent.*

The results from [KPb] suggest that double covering of quadric ramified in octic with 20 nodal singular points will also have strictly unipotent monodromy of the singular fiber at zero of its Landau–Ginzburg model. Indeed this double covering is nothing else but a three dimensional quartic deformation and its monodromy was computed in [KPb]. We extract categorical information from this common phenomenon — strict unipotency of monodromy in following theorems and conjectures.

Let us denote by $H(LG(X), \mathcal{F})$ the hypercohomologies of the perverse sheaf of vanishing cycles on the Landau–Ginzburg model. $H(LG(X), \mathcal{F})$ measure cohomologies of X and the monodromy of $LG(X)$ — see [GKR] and [Ka09].

Theorem 5.4. *Let X be a smooth Fano variety. Let $LG(X)$ be its Landau–Ginzburg model (in particular, HMS for X and $LG(X)$ holds). Then the Hochschild homology of Fukaya–Seidel category of $LG(X)$ is $H(LG(X), \mathcal{F})$.*

Proof. It follows from [KKP08]. □

According to Homological Mirror Symmetry the Hochschild homology of Fukaya–Seidel category of Landau–Ginzburg model are isomorphic to Hochschild homology of category $D^b(X)$.

Combining results from Subsections 5.1 and 5.2 with conifold transition change described in [Ka09] we get

Proposition 5.5. *The Hochschild homology of category $D^b(X)$ of Artin–Mumford example and of resolved V_{10} with 10 singular points are isomorphic.*

Proof. This follows from a direct calculations of the cohomology of resolved V_{10} with 10 singular points. \square

In fact these homology look like cohomologies of a projective space.

Using above analysis of monodromy of Landau–Ginzburg models of Artin–Mumford example, V_{10} with 10 singular points, of double covering of quadric ramified in octic with 20 nodal singular points, and of double solid with ramification in a sextic with 35 singular points (see [KPb]) we arrive at

Conjecture 5.6. *The categories $D^b(X)$ of the Artin–Mumford example, of V_{10} with 10 singular points, of double covering of quadric ramified in octic with 20 nodal singular points, and of double solid with ramification in a sextic with 35 singular points contain category of a nodal Enriques surface as a direct summand.*

Remark 5.7. While this paper was being written Kuznetsov and Ingalls, familiar with our work, proved above conjecture for Artin–Mumford example — [IK10]. The first two authors are collaborating with A. Kuznetsov in order to prove this conjecture for V_{10} with 10 singular points.

In the next section we look at the above observations from prospective of theory of spectra of category.

6. SPECTRUM, ENHANCED SPECTRUM AND APPLICATIONS

6.1. Classical Spectrum. In this subsection we review the notions of spectra and gaps following [BFK10].

Noncommutative Hodge structures were introduced in [KKP08], as a means of bringing the techniques and tools of Hodge theory into the categorical and noncommutative realm. In the classical setting much of the information about an isolated singularity is recorded by means of the Hodge spectrum, a set of rational eigenvalues of the monodromy operator. A categorical analogue of this Hodge spectrum appears in the works of Orlov and Rouquier [Or08], [Ro08]. Let us call this *the Orlov spectrum*. Recent work in the manuscript [BFK10], suggests an intimate connection with the classical singularity theory.

Let us recall the definitions of the Orlov spectrum and discuss some of the main results in [BFK10]. Let \mathcal{T} be a triangulated category. For any $G \in \mathcal{T}$ denote by $\langle G \rangle_0$ the smallest full subcategory containing G which is closed under isomorphisms, shifting, and taking finite direct sums and summands. Now inductively define $\langle G \rangle_n$ as the full subcategory of objects, B , such that there is a distinguished triangle, $X \rightarrow B \rightarrow Y \rightarrow X[1]$, with $X \in \langle G \rangle_{n-1}$ and $Y \in \langle G \rangle_0$.

Definition 6.1. Let G be an object of a triangulated category \mathcal{T} . If there is an n with $\langle G \rangle_n = \mathcal{T}$, we set

$$t(G) = \min \{n \geq 0 \mid \langle G \rangle_n = \mathcal{T}\}.$$

Otherwise we set $t(G) = \infty$. We call $t(G)$ the *generation time* of G . If $t(G)$ is finite, we say that G is a *strong generator*. The *Orlov spectrum* of \mathcal{T} is the union of all possible generation times for strong generators of \mathcal{T} . The Rouquier dimension is the smallest number

in the Orlov spectrum. We say that a triangulated category \mathcal{T} has a *gap* of length s if a and $a + s$ are in the Orlov spectrum but r is not in the Orlov spectrum for $a < r < a + s$. We denote the maximum (finite) gap of the Orlov spectrum of \mathcal{T} by $\text{Gap}(\mathcal{T})$.

The following 3 conjectures are from [BFK10].

Conjecture 6.2. *If X is a smooth variety then any gap of $D^b(X)$ is at most the Krull dimension of X .*

Conjecture 6.3. *The maximal gap in Orlov's spectrum is a birational invariant.*

In particular, this conjecture says that if X is smooth projective rational threefold then gap of $D^b(X)$ is equal to 1.

We now apply theory of gaps to the observations from the previous sections. First we formulate:

Conjecture 6.4. *Let X be a smooth algebraic surface. Then $h^{2,0}(X) = 0$ is equivalent to $\text{Gap}(D^b(X)) = 1$.*

Combining this conjecture with Conjecture 5.6 we get

Conjecture 6.5. *The gap of the category $D^b(X)$ for the Artin–Mumford example, of V_{10} with 10 singular points, of the double covering of quadric ramified in octic with 20 nodal singular points, and of the double solid with ramification in a sextic with 35 singular points is equal to 1.*

In other words the gap of Orlov spectra is too weak of a categorical invariant to distinguish the rationality of these examples. In the next section we introduce more advanced Noether–Lefschetz spectra.

6.2. Enhanced Noether–Lefschetz Spectra. Let \mathcal{T} be a triangulated category and let $HH_*(\mathcal{T})$ be its Hochschild homology.

Definition 6.6. We denote by $NL(\mathcal{T})$ the ordered collection of sets over $HH(\mathcal{T})$ defined as follows. For any subspace I in $HH_*(\mathcal{T})$ we consider the DG subcategory $\text{Ann}(I)$ in \mathcal{T} — the annihilator of I . The set $\text{Spec}(\text{Ann}(I))$ is the set of generators of \mathcal{T} in the DG subcategory $\text{Ann}(I)$. We denote the maximum gap of $\text{Spec}(\text{Ann}(I))$ over all subsets I by $NL\text{Gap}(\mathcal{T})$ (see Figure 13).

Clearly $\text{Spec}(\mathcal{T})$ embeds in the set $(I, \text{Spec}(\text{Ann}(I)))$ but the behavior of the gaps in $NL(\mathcal{T})$ is much more complex. For more examples see [FKb].

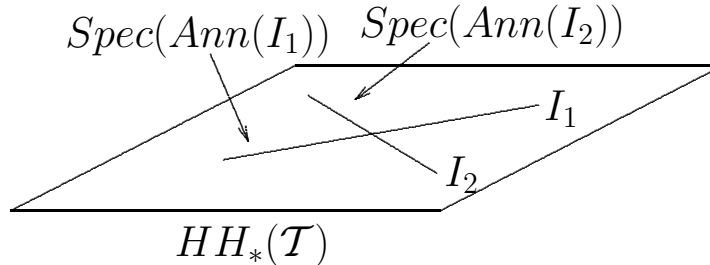


FIGURE 13. NL Spectra.

We make the following:

Conjecture 6.7. *Let X be a 3-dimensional smooth projective variety. If X is rational then the gaps in $NL(D^b(X))$ are equal to 1.*

The above conjecture suggests a new invariant of rationality. It is based on our studies of Landau–Ginzburg models from previous sections. Theorem 5.1 together with HMS suggests that $NL(D^b(X))$ are completely determined by monodromy and vanishing cycles of Landau–Ginzburg models, see Table 1. Still it is possible that $NL(D^b(X))$ has all gaps equal to one and X is not rational.

A category \mathcal{T}	$NLSpec(\mathcal{T})$
$D^b(X)$	$NLSpec(\mathcal{T}) \subset Spec_{dg-gr}(HH^*(D^b(X))) \times Spec(\mathcal{T})$
$FS(LG(X))$	$NLSpec(\mathcal{T}) \subset Spec_{dg-gr}(H^*(LG(X), \mathcal{F})) \times Spec(\mathcal{T})$

TABLE 1. HMS and Noether–Lefschetz spectra.

In what follows we give conjectural examples of 3-dimensional varieties which have gaps equal to 1 in $Spec(D^b(X))$ and have gaps equal to two or higher in $NL(D^b(X))$. Following Conjecture 6.3 Homological Mirror Symmetry, and examples in Section 5 we make

Conjecture 6.8. *In all examples: Artin–Mumford example, V_{10} with 10 singular points, double covering of quadric ramified in octic with 20 nodal singular points, and double solid with ramification in a sextic with 35 singular points $NLGap(D^b(X))$ is equal to two or higher.*

This conjecture is based on the fact that Landau–Ginzburg models for Artin–Mumford example, for V_{10} with 10 singular points, for double covering of quadric ramified in octic with 20 nodal singular points and for double solid with ramification in a sextic with 35 singular points have the same monodromies — see also [KPb].

We record all our findings and conjectures in Table 2.

Remark 6.9. It is quite possible that derived categories of the Artin–Mumford example and of V_{10} are related via deformation in which case equalities of spectra is not surprising.

Remark 6.10. The considerations in the last two sections suggest a strong correlation between spectra, monodromy and walls in moduli spaces of stability conditions. We pose the following two questions:

Question 1. Does Noether–Lefschetz spectra define a stratification on the moduli space of stability conditions?

Question 2. Are classical Noether–Lefschetz loci connected to this stratification?

Remark 6.11. Artin–Mumford example is an example of a conic bundle. We expect that technique discussed here will lead to many examples of conic bundles for which the gap of Orlov’s spectrum is equal to one and their nonrationality can be established using gaps in Noether–Lefschetz spectra.

A Fano variety X	$D^b(X)$ and $HH_0(X)$	Gap ($D^b(X)$)	NLGap (X)
A double covering of \mathbb{P}^3 ramified in K3 surface with 10 nodal singular points (Artin–Mumford variety).	$D^b(X) = \langle D^b(E), E_1, \dots, E_{10} \rangle$, where E is a nodal Enriques surface. $\dim(HH_0(X)) = 4$	1	≥ 2
Double covering $\begin{array}{c} V_{10} \\ \downarrow 2:1 \\ V_5 \end{array}$	$D^b(X) = \langle D^b(E), \dots \rangle$, where E is a nodal Enriques surface. $\dim(HH_0(X)) = 4$	1	≥ 2
\mathbb{P}^3	$\dim(HH_0(X)) = 4$	1	1
Sextic double solid with 35 nodal singular points.	$D^b(X) = \langle D^b(E), \dots \rangle$, where E is a nodal Enriques surface. $\dim(HH_0(X)) = 4$	1	≥ 2
Double covering of quadric ramified in octic with 20 nodal singular points.	$D^b(X) = \langle D^b(E), \dots \rangle$, where E is a smooth Enriques surface. $\dim(HH_0(X)) = 4$	1	≥ 2

TABLE 2. Summarizing conjectures.

REFERENCES

- [AM72] M. Artin, D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. (3) **25** (1972) 75–95.
- [AMG96] P. Aspinwall, D. Morrison, M. Gross, *Stable Singularities in String Theory*, Commun. in Math. Physics, **178** No. 1 (1996) 115–134; arXiv: hep-th/9503208.
- [Be83] A. Beauville, *Variétés rationnelles et unirationnelles*, Lect. Notes in Math. **997** (1983) 16–33.
- [BFK10] M. Ballard, D. Favero, L. Katzarkov, *The Orlov Spectrum: Gaps and Bounds*, arXiv:1012.0864.
- [BT82] R. Bott, L. Tu, *Differential forms in algebraic topology*, GTM, Vol. **82**, Springer–Verlag 1982.
- [Co83] F. Cossec, *Reye congruences*, Trans. AMS, **280** No. 2 (1983) 737–751.
- [CP07] I. Cheltsov, J. Park, *Two remarks on sextic double solids*, J. Number Theory **122** No. 1 (2007) 1–12; I. Cheltsov, *Double spaces with isolated singularities*, Sb. Math. **199**, No. 2 (2008) 291–306.
- [En99] S. Endrass, *On the divisor class group of double solids*, Manuscr. Math. **99** No. 3 (1999) 341–358.
- [FKa] D. Favero, L. Katzarkov, *A Categorical Griffiths Residue Map and the Hodge Conjecture*, in preparation.
- [FKb] D. Favero, L. Katzarkov, *Noether–Lefschetz Spectra and Algebraic cycles*, in preparation.
- [GKKN] R. Garavuso, L. Katzarkov, M. Kreuzer, A. Noll, *Super Landau–Ginzburg mirrors and algebraic cycles*, arXiv:1101.1368.
- [GKR] M. Gross, L. Katzarkov, H. Rudat, *Homological Mirror Symmetry and Hypercohomologies*, in preparation.
- [Gr68] A. Grothendieck, *Le groupe de Brauer I, II, III*, in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. in Pure Math. 3, p. 46–66, 67–87, 88–188, North Holland, Amsterdam (1968).

- [Hi78] F. Hirzebruch, *Topological methods in algebraic geometry*, GMW, Vol. **131**, Springer-Verlag (1978).
- [HT84] J. Harris, L. Tu, *On Symmetric and Skew-symmetric Determinantal Varieties*, *Topology* **23** (1984) 71–84.
- [IK10] C. Ingalls, A. Kuznetsov, *On nodal Enriques surfaces and quadric double solids*, arXiv:1012.3530.
- [IP99] V. Iskovskikh, Yu. Prokhorov, *Fano varieties*, *Encyclopaedia Math. Sci.* 47 (1999) Springer, Berlin.
- [IKS] A. Iliev, L. Katzarkov, E. Scheidegger, *4 dimensional cubics, Noether–Lefschetz loci and gaps*, in preparation.
- [Is80] V. A. Iskovskikh, *Birational automorphisms of three-dimensional algebraic varieties*, *J. Sov. Math.* **13** (1980) 815–868.
- [JLP82] T. Jozefiak, A. Lascoux, P. Pragacz, *Classes of determinantal varieties associated with symmetric and skew-symmetric matrices*, *Math. USSR Izv.* **18** (1982) 575–586.
- [Ka09] L. Katzarkov, *Homological Mirror Symmetry and Algebraic cycles*, *Progress in Math* 271. Birkhauser, 2009, Conf. in honour of C. Boyer.
- [KKP08] L. Katzarkov, M. Kontsevich, T. Pantev, *Hodge theoretic aspects of mirror symmetry*, in *From Hodge theory to integrability and TQFT: tt*-geometry* (R. Donagi and K. Wendland, eds.), *Proc. Symposia in Pure Math.*, Vol. 78, AMS, Providence, RI, 2008, p. 87–174.
- [KK] L. Katzarkov, G. Kerr, *Stability, Spectra and wall crossings*, in preparation.
- [KNS] L. Katzarkov, A. Nemethi, D. Stepanov *Four dimensional cubics spectra and monodromy*, in preparation.
- [KKP] L. Katzarkov, M. Kontsevich, T. Pantev, *Hodge theoretic aspects of mirror symmetry 2*, in preparation.
- [KMZ] L. Katzarkov, G. Mikhalkin, I. Zharkov, *Tropical Hodge Theory*, in preparation.
- [KPa] L. Katzarkov, V. Przyjalkowski, *New categorical invariants and applications*, in preparation.
- [KPb] L. Katzarkov, V. Przyjalkowski, *Landau–Ginzburg models with many potentials and applications*, in preparation.
- [Kuz08] A. Kuznetsov, *Derived categories of Fano threefolds*, arXiv:0809.0225.
- [Kuz09] A. Kuznetsov, *Hochschild homology and semiorthogonal decompositions*, arXiv:0904.4330.
- [Ol94] C. Oliva, *Algebraic cycles and Hodge theory on generalized Reye congruences*, *Compos. Math.* **92** No. 1 (1994) 1–22.
- [Or08] D. Orlov, *Remarks on generators and dimensions of triangulated categories*, arXiv:0804.1163.
- [Prz09a] V. Przyjalkowski, *Weak Landau–Ginzburg models for smooth Fano threefolds*, arXiv:0902.4668.
- [Prz09b] V. Przyjalkowski, *Hodge numbers of Fano threefolds via Landau–Ginzburg models*, arXiv:0911.5428.
- [Ro08] R. Rouquier, *Dimensions of triangulated categories*, *J. K-Theory* 1 (2008), no. 2, 193–256.
- [Za77] A. Zagorskii *Three-dimensional conical fibrations*, *Math. Notes* **21** (1977) 420–427.

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